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Optimization of axisymmetric membrane shells☆

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Abstract

Problems of the joint optimization of the shape and distribution along the meridian of the thickness of membrane shells of revolution under the action of axisymmetric loads are considered, taking account of the constraints concerning the strength of the shell and the volume of its cavity. General formulations of problems of the optimal design of shells of revolution are given and the optimal shape of a shell and the corresponding thickness distribution are investigated. Results of the exact solution of problems of the optimal design of closed shells of revolution when there is an internal pressure are presented. The simultaneous introduction of two control functions, describing the shape of the shell and the distribution of its thickness, not only ensures a substantial reduction in the mass of a shell but also leads to significant mathematical simplifications, which enable the solution of the optimization problem being considered to be obtained in an analytical form.

A considerable number of papers (see Refs 1,2, for example), which mainly refer to the case of a fixed shape of the neutral surface and a thickness distribution which is optimizable along a given surface, have been devoted to problems of the optimal design of axisymmetric shells and composite shells of revolution. The search for the optimal shape of the middle surface of a shell for a specified thickness distribution presents considerable difficulties even in the case of constant thickness and leads to problems, the solution of which can only be obtained using numerical methods (see Refs 3,4).

1. Basic relations in the problem of optimizing the shape of a shell of revolution

Consider a shell which has the shape of a surface of revolution, the axis of which coincides with the x-axis (Fig. 1). The position of the meridian plane is specified by the angle θ, which is measured from a certain fixed plane and the alignment of the parallel circle is defined by the angle φ between the normal to the surface and the axis of rotation, $r = r(x)$ is the radius of the parallel circle, which determines the distance from a point on the neutral surface of the shell to its axis of rotation and $0 \le x \le \mathrm{L}$, where L is the specified length of the shell. The quantities $r(0) = r_1$ and $r(L) = r_2$, which correspond to the ends of the shell, are assumed to be given and to satisfy the inequalities $r_1 \ge 0$, $r_2 \ge 0$, $r_1 \le r_2$. The meridian plane and the plane which is perpendicular to the meridian are the planes of principal curvatures of the surface of the shell at the point being considered. The corresponding radii of the principal curvatures are r_φ and r_θ.

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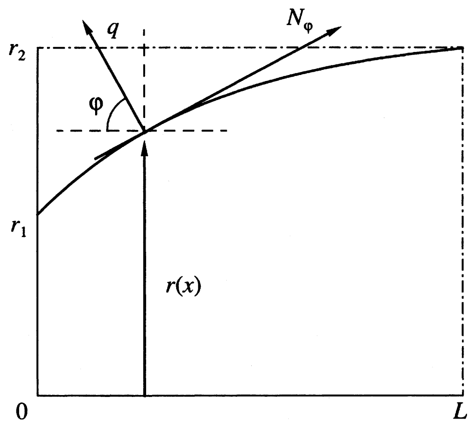

Fig. 1.

The thickness distribution $h = h(x)$ is assumed to satisfy the well-known condition in the theory of thin elastic shells

$$h(x) \le h_m = \max_x h(x) \ll r_m; \quad r_m = \min\{\min_x r_\varphi(x), \min_x r_\theta(x)\} \tag{1.1}$$

The minima with respect to x in (1.1) are defined in the interval $[0,L]$ and the external operation min in front of the braces denotes the search for the smaller of the two quantities.

The shell is loaded with a constant internal pressure q and distributed shear forces on its ends: $x = 0$ and $x = L$, the resultants of which $R_1(x = 0)$ and $R_2(x = \mathrm{L})$ are directed along the axis of the shell (Fig. 1). If the ends $x = 0$ and $x = L$ of the shell are fitted with circular end plates, then $R_1 = \pi r_1^2 q$, $R^2 = \pi r_2^2 q$. If the shell has poles $r(0) = r_1 = 0$ and $r(L) = r_2 = 0$ at its ends as, for example, in the case of closed ellipsoidal shells of revolution, then $R_1 = 0$, $R_2 = 0$.

The equation of equilibrium of an element of a membrane shell, written for the direction normal to the neutral surface of the shell and which relates the magnitudes of the normal membrane stresses N_φ, N_θ, has the form[1,5,6]

$$N_\varphi / r_\varphi + N_\theta / r_\theta = q \tag{1.2}$$

The radii of curvature r_φ and r_θ are related to the radius of the transverse cross-section of the shell by the relations (everywhere hereafter a derivative with respect to x is denoted by a prime)

$$r_\theta = r\sqrt{1 + r'^2}, \quad r_\varphi = -(1 + r'^2)^{3/2} / r'' \tag{1.3}$$

The equilibrium equation for the cut off part of the shell $x < L$ in the axial direction can be written as follows

$$\frac{2\pi r}{\sqrt{1 + r'^2}} N_\varphi = R_1 + \pi q (r^2 - r_1^2) \tag{1.4}$$

The expressions for the magnitudes of the normal membrane stresses follow from Eqs. (1.2)–(1.4).

For the normal stresses arising in the shell we have

$$\sigma_\varphi = N_\varphi / h, \quad \sigma_\theta = N_\theta / h \tag{1.5}$$

For a fixed shape of the middle surface of the shell $r = r(x)$ and a specified thickness distribution $h = h(x)$, the volume of the material of the shell is given by the formula

$$J = 2\pi \int_0^L r h \sqrt{1 + r'^2}\, dx \tag{1.6}$$

The optimization problem being considered consists of searching for the shape of the shell $r = r(x)$ and its thickness distribution $h = h(x)$ which make the volume of the shell material a minimum when the strength condition

$$\max\{\sigma_\varphi, \sigma_\theta\} \le \sigma_* \tag{1.7}$$

the boundary conditions

$$r(0) = r_1, \quad r(L) = r_2 \tag{1.8}$$

and the geometrical constraints imposed on the volume of the shell cavity

$$\pi\int_0^L r^2 dx = V_0 \tag{1.9}$$

are satisfied, where σ_*, r_1, r_2, L and V_0 are specified positive constants. The strength condition can be written in the form of (1.7) in the case of membrane shells of revolution made of brittle and quasibrittle materials. At the same time the constant strength of the material σ_* (the reduced maximum stress) is determined using known values of the constant quasibrittle strength of the material K_{1c} and the maximum admissible length of the initial normal tensile cracks, l_m, which arise in the manufacture of a shell or as a result of its use[7,8]

2. The search for the shape and thickness distribution of an optimal shell

We will consider the problem of the optimal design of a closed axisymmetric shell with rigid end plates at $x = 0$ and $x = L$ that is acted upon by an internal pressure q. We have

$$R_1 = \pi r_1^2 q, \quad R_2 = \pi r_2^2 q \tag{2.1}$$

When constructing the optimal solution, we shall assume that, in inequality (1.7), the sign of strict equality holds over the whole of the interval $x \in [0, L]$:

$$\sigma_\varphi = N_\varphi / h = \sigma_*, \quad \sigma_\theta \le \sigma_* \tag{2.2}$$

or

$$\sigma_\theta = N_\theta / h = \sigma_*, \quad \sigma_\varphi \le \sigma_* \tag{2.3}$$

We shall initially assume that conditions (2.2) are satisfied over the whole of the interval $[0, L]$, that is, the constraint imposed on the meridian stress σ_* is active. It then follows from relations (1.4), (2.2) and (2.1) that the optimal thickness distribution $h = h_\varphi = h(x)$ and the corresponding optimal shape of the shell $r = r(x)$ are related by the equality

$$h_\varphi = \frac{qr}{2\sigma_*}\sqrt{1 + r'^2} \tag{2.4}$$

The subscript φ on the distribution of the thickness h denotes that, when relation (2.4) is satisfied, the meridian stress σ_φ is critical and the peripheral stress σ_θ is assumed to satisfy the inequality $\sigma_\theta \le \sigma_\varphi$.

Using relations (1.2), (1.4) (1.9) and (2.4), we arrive at the following representations for the minimized volume of the shell material (the functional of the problem) and for the augmented Lagrange functional, which takes account of the given volume of the shell cavity (1.9)

$$J = 2\pi\int_0^L r h_\varphi \sqrt{1 + r'^2} dx = \frac{\pi q}{\sigma_*}\int_0^L r^2 (1 + r'^2) dx \tag{2.5}$$

$$J^a = J - \lambda V_0 = \frac{\pi q}{\sigma_*}\int_0^L \{(rr')^2 + \beta r^2\} dx, \quad \beta = 1 - \frac{\lambda\sigma_*}{q} \tag{2.6}$$

where λ is the Lagrange multiplier corresponding to condition (1.9). The necessary condition for an extremum of the functional (2.6) (Euler's equation) under the assumption that $r(x) \neq 0$ when $x \in [0, L]$ has the form

$$rr'' + r'^2 - \beta = 0 \tag{2.7}$$

In this case, the search for the shape of the optimal shell reduces to solving of the following boundary value problem

$$(r^2)'' = 2\beta, \quad r(0) = r_1, \quad r(L) = r_2 \tag{2.8}$$

We now introduce the notation

$$\Delta_{\pm} = (r_2^2 \pm r_1^2)/L^2$$

The solution of problem (2.8) has the form

$$r^2 = \beta x^2 + ax + r_1^2, \quad a = L(\Delta_- - \beta) \tag{2.9}$$

The optimal thickness distribution $h\varphi = h_\varphi(x)$, corresponding to the optimal shape which has been found, is obtained using relations (2.4) and (2.9) and is written in the form

$$h_\varphi = \frac{q}{2\sigma_*}\sqrt{\beta(1+\beta)x^2 + a(1+\beta)x + a^2/4 + r_1^2} \tag{2.10}$$

The Lagrange multiplier λ and the parameter β are found using expression (2.9), the isoperimetric condition (1.9) imposed on the volume of the shell cavity and formula (2.6) which relates the quantities λ and β. We have

$$\beta = -\alpha, \quad \alpha = \frac{6V_0}{\pi L^3} - 3\Delta_+, \quad \lambda = \frac{q}{\sigma_*}(1-\beta) = \frac{q}{\sigma_*}(1+\alpha) \tag{2.11}$$

It follows from relations (2.5) and (2.9) that, in the case of the scheme (2.9)–(2.11) being considered, the minimum volume of shell material is given by the formula

$$J = J_\varphi = \frac{q}{\sigma_*}V_0 + \frac{\pi q L}{\sigma_*}\left(\frac{\beta^2 L^2}{3} + \frac{\beta L a}{2} + \frac{a^2}{4}\right) = \alpha(\alpha+2) + 6\Delta_+ + 3\Delta_-^2 \tag{2.12}$$

The domain of variation of the parameters of the problem V_0, L, r_1, r_2, which ensures the existence of the solution of the form being considered, is established using the strength condition

$$\sigma_\theta(x) \le \sigma_\varphi(x) = \sigma_*, \quad 0 \le x \le L$$

which we convert to the inequality

$$\sigma_\theta = \sigma_*(2 + r'^2 + \beta)(1 + r'^2)^{-1} \le \sigma_*, \quad 0 \le x \le L$$

The solution of this inequality leads to the constraint

$$\alpha = -\beta = \frac{6V_0}{\pi L^3} - 3\Delta_+ \ge 1 \tag{2.13}$$

The optimal shapes of the shells $r(x)$ and their thickness distributions $h = h_\varphi(x)$ which have been found are shown in Fig. 2 for $\alpha = 5, 2, 1$ in the case when

$$r(0) = L/2, \quad r(L) = L \tag{2.14}$$

It can be seen from relations (2.9) and (2.10) and the graphs that the optimal shell when $\alpha \ge 1$ has the shape of the part of an ellipsoid of revolution located between the cross-sections $x = 0$ and $x = L$. The corresponding thickness distribution has its maximum values close to the ends of the shell and, when $\alpha \to 1$, the thickness of the shell tends to a constant value $h_\varphi(x) = (q/(2\sigma_*))\sqrt{a^2/4 + r_1^2}$.

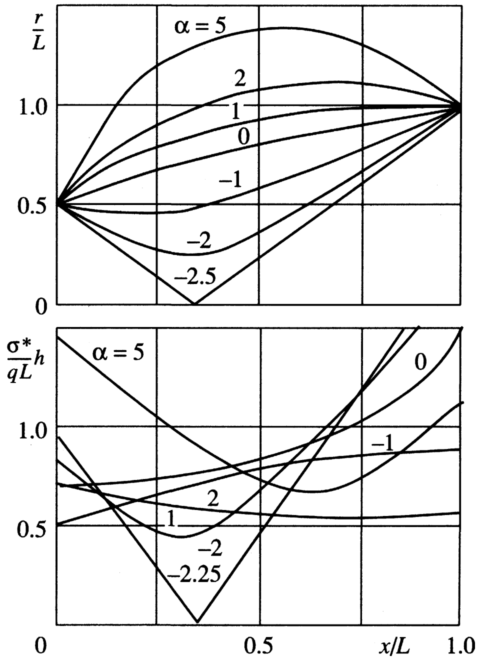

Fig. 2.

When $r(0) = r(L)$, the shape of the shell and the thickness distribution are symmetrical about to the cross-section $x = L/2$. If $r(0) = r(L) = 0$, the optimal shell takes the shape of an oblate ellipsoid of revolution

$$\tilde{r}^2 + \alpha\left(\tilde{x} - \frac{1}{2}\right)^2 = \frac{\alpha}{4}, \quad \tilde{r} = \frac{r}{L}, \quad \tilde{x} = \frac{x}{L}$$

and the maximum value of the thickness, shown in Fig. 3, is attained at the poles of the ellipsoid. When $\alpha \to 1$, the optimal shell acquires a spherical shape and its thickness becomes constant. The dependence of the mass of the shell on the parameter α when $\alpha \geq 1$ and $r(0) = r(L) = 0$ is represented in dimensionless form in Fig. 4.

We will now consider another case when conditions (2.3) are satisfied over the whole of the interval $[0, L]$, that is, the constraint imposed on the peripheral stress σ_θ is active. According to relations (1.2), (1.4), (2.1) and (2.3), the optimal thickness distribution $h = h_\theta = h_\theta(x)$ and the corresponding optimal shape of the shell $r = r(x)$ are connected by

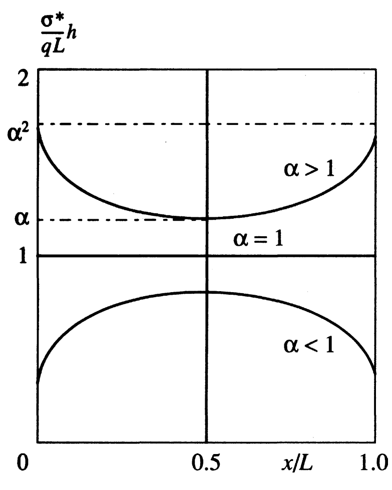

Fig. 3.

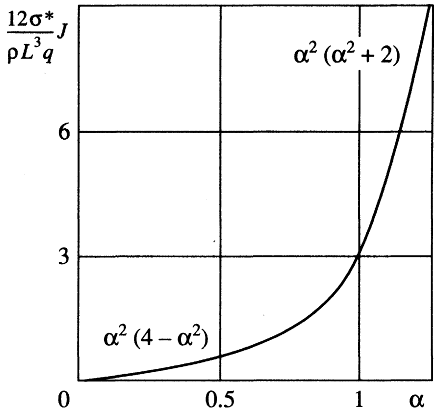

Fig. 4.

the relation

$$h_\theta = \frac{qr}{2\sigma_*}\left(\frac{1}{2}rr'' + r'^2 + 1\right)(1 + r'^2)^{-1/2} \tag{2.15}$$

The subscript θ on the distribution of the thickness h denotes that the peripheral stress σ_θ is critical when relation (2.14) is satisfied and the meridian stress σ_φ is assumed to satisfy the inequality $\sigma_\varphi \le \sigma_\theta$.

Using relations (1.6), (1.9) and (2.15), we can represent the expression for the minimized volume of the shell material and for the augmented Lagrange functional, which allows for the specification of the volume of the cavity of the shell, in the following manner

$$\begin{aligned} J &= 2\pi\int_0^L rh_\theta\sqrt{1 + r'^2}dx = \frac{\pi q}{\sigma_*}\int_0^L r^2(rr'' + 2r'^2 + 2)dx \\ J^a &= J - \lambda V_0 = \frac{\pi q}{\sigma_*}\int_0^L [r^2(rr'' + 2r'^2) - \gamma r^2]dx, \quad \gamma = \frac{\lambda\sigma_*}{q} - 2 \end{aligned} \tag{2.16}$$

where λ is the Lagrange multiplier corresponding to condition (1.9).

The Euler equation

$$rr'' + r'^2 - \gamma = 0 \tag{2.17}$$

written for the augmented functional, which is defined by the second equality of (2.16), leads, jointly with relations (1.8) when account is taken of the inequality $r(x) \ne 0$ when $x \in (0, L)$, to the following boundary-value problem for determining the optimal shape of a shell $r(x)$

$$(r^2)'' = 2\gamma, \quad r(0) = r_1, \quad r(L) = r_2 \tag{2.18}$$

The optimal scheme, which is found from relations (2.15), (2.17) and (2.18), has the form

$$r^2 = \gamma x^2 + bx + r_1^2, \quad b = L(\Delta_- - \gamma) \tag{2.19}$$

$$h_\theta = \frac{q}{2\sigma_*}[\chi + (1 + \gamma)(\gamma x^2 + bx + r_1^2)\chi^{-1}], \ \chi = \sqrt{\gamma(1 + \gamma)x^2 + (1 + \gamma)bx + b^2/4 + r_1^2} \tag{2.20}$$

The parameters γ and λ are defined as

$$\gamma = -\frac{12V_0}{\pi L^3} + 6\Delta_+, \quad \lambda = 2\frac{q}{\sigma_*}\left(1 - \frac{3V_0}{\pi L^3} + \frac{3}{2}\Delta_+\right) \tag{2.21}$$

For the volume of the shell material corresponding to the given scheme, we have the expression

$$J = J_\theta = \frac{q}{\sigma_*}(2+\gamma)V_0 + \frac{\pi q L}{4\sigma_*}\left(\frac{\gamma^2 L^2}{3} + \frac{\gamma b L}{2} + \frac{b^2}{4}\right) = \alpha(4-\alpha) + 6\Delta_+(2-a) + 3\frac{\Delta_-}{L^2} \quad (2.22)$$

The domain of variation of the parameters which ensures the existence of the optimal solution in the form of (2.19)–(2.22) is established using the stability condition

$$\sigma_\varphi \le \sigma_\theta = \sigma_*, \quad 0 \le x \le L$$

The inequality $\sigma_\varphi \le \sigma_\theta$ leads to the relation

$$rr'' + r'^2 + 1 \ge 0 \quad (2.23)$$

and the condition $\sigma_\varphi \le \sigma_*$, using relations (1.4), (1.5) and (2.14), is written in the form

$$\sigma_\varphi = \sigma_* \frac{1 + r'^2}{rr'' + 2r'^2 + 2} \le \sigma_* \quad (2.24)$$

It is easily noted, using inequality (2.23), that the denominator in relation (2.24) is positive and the following constraint, imposed on the parameters of the problem, therefore results from the inequality (2.24):

$$\alpha = -\gamma = \frac{6V_0}{\pi L^3} - 3\Delta_+ \le 1 \quad (2.25)$$

A second constraint follows from the condition

$$r(x) \ge 0, \quad 0 \le x \le L$$

and has the form

$$\alpha \ge \alpha_0, \quad \alpha_0 = -(r_1 + r_2)^2/L^2 \quad (2.26)$$

The curvature of the optimal shell in the direction of the meridian vanishes, that is,

$$|r_\varphi| = \infty, \quad 0 \le x \le L$$

for values of the parameter α which are determined from the condition

$$r'' = 0, \quad 0 \le x \le L$$

and are equal to $\alpha_1 = -(r_1 - r_2)^2/L^2$.

When $\alpha = \alpha_1$, the optimal shell has conical shape

$$r(x) = r_1 + x(r_2 - r_1)/L, \quad 0 \le x \le 1 \quad (2.27)$$

and, when $\alpha = \alpha_0$, it is described by the two conical surfaces

$$\begin{aligned} r &= -(r_1 + r_2)x/L + r_1 \quad \text{When} \quad 0 \le x \le x_0 = r_1 L/(r_1 + r_2) \\ r &= (r_1 + r_2)x/L - r_1 \quad \text{When} \quad x_0 \le x \le L \end{aligned} \quad (2.28)$$

corresponding to the broken line in the upper part of Fig. 2. We note that the shape (2.27) is the limiting shape for the solutions being considered when $\alpha_0 \le \alpha \le 1$.

The shell of conical form (2.27) partitions the whole family of the optimal shell shapes which have been constructed into biconvex shells of positive Gaussian curvature ($k = (r_\varphi r_\theta)^{-1} > 0$) when $\infty > \alpha > \alpha_1$ and convex-concave shells of negative Gaussian curvature ($k < 0$) when $\alpha_0 < \alpha < \alpha_1$

The optimal shapes of the shells $r = r(x)$ and their thickness distributions $h = h_\theta(x)$ in the case when equalities (2.14) are satisfied are shown in Fig. 2 for $\alpha = 1, 0, -1, -2$ and -2.25. The optimal shape and thickness distribution, corresponding to the limiting value $\alpha = 1$ are identical to the corresponding curves obtained from relations (2.9) and

(2.10) when $\alpha = 1$.It can be seen from the upper part of Fig. 2 that, as α increases, the curvature of the shell changes sign.

When $r(0) = r(L)$, the optimal shape of the shell and the corresponding thickness distribution are symmetrical about the transverse cross-section $x = L/2$. If $r(0) = r(L) = 0$, then the dependence of the mass of the shell on the dimensionless parameter α is given by the expression $(12\sigma_*/(\pi q L_3))\,J = \alpha^2(4 - \alpha^2)$, which is shown in Fig. 4 for $\alpha \le 1$. The optimal shell corresponding to this case has the shape of a prolate ellipsoid of revolution and the maximum values of the thickness are attained at the equator ($x = L/2$).

3. Some properties of the optimal solution

It was assumed when constructing the optimal solution that, in relation (1.7), the sign of strict equality holds over the whole of the interval $[0, L]$, that is, relations (2.2) and (2.3) are satisfied. It is proved below that, if a solution exists which satisfies the above mentioned relations (the strict equalities of the attainment of the limit values), then it will be the optimal solution.

We will prove this assertion by an indirect method. We will assume that the optimal solution $(r_{\text{opt}}(x), h_{\text{opt}}(x))$ satisfies the strict inequality in (1.7), or

$$h_{\text{opt}} > N_\varphi/\sigma_*, \quad h_{\text{opt}} > N_\theta/\sigma_* \tag{3.1}$$

in a certain segment $[x_1, x_2]$ $(0 \le x_1 < x_2 \le L)$. The optimal solution $(r_{\text{opt}}(x), h_{\text{opt}}(x))$ which is being considered is assumed to satisfy conditions (2.2) and (2.3) in the remaining segments $0 \le x < x_1$ and $x_2 < x < L$ of the interval $[0, L]$, that is, it is assumed that the sign of strict equality holds in relation (1.7). In this case, it is possible to construct an admissible design $(\hat{r}(x), \hat{h}(x))$ in the following form

$$\hat{r} = r_{\text{opt}}, \quad \hat{h} = \begin{cases} h_{\text{opt}} & \text{if } 0 \le x < x_1, \quad x_2 < x \le L \\ N_\varphi(r_{\text{opt}})/\sigma_* < h_{\text{opt}} \text{ or } \hat{h} = N_\theta(r_{\text{opt}})/\sigma_* < h_{\text{opt}}, & \text{if } x_1 \le x \le x_2 \end{cases} \tag{3.2}$$

The design $(\hat{r}, \hat{h})$ is admissible, since the distributions $\hat{r}(x)$, $\hat{h}(x)$ which have been presented satisfy the stability condition (1.7) and the isoperimetric constraint imposed on the volume of the shell cavity (1.9). Note that the admissible design (3.2) which has been presented satisfies relation (1.7) with the sign of strict equality over the whole of the interval $[0, L]$. Hence, the admissible solution constructed is an equally strength design, and for it we shall have

$$J(\hat{r}, \hat{h}) = I_{0, x_1} + 2\pi \int_{x_1}^{x_2} r_{\text{opt}} \hat{h} \sqrt{1 + r'^2_{\text{opt}}}\, dx + I_{x_2, L} < I_{0, L} = J(r_{\text{opt}}, h_{\text{opt}}) \tag{3.3}$$

where

$$I_{a, b} = 2\pi \int_a^b r_{\text{opt}} h_{\text{opt}} \sqrt{1 + r'^2_{\text{opt}}}\, dx$$

The inverse equality established in this manner

$$J(r_{\text{opt}}, h_{\text{opt}}) > J(\hat{r}, \hat{h})$$

proves the assertion that relation (1.7) with the sign of strict equality holds over the whole of the interval $[0, L]$ in the case of the optimal solution when it exists.

Thus, when $\alpha \ge 1$, the shape of the optimal shell is specified by a biconvex surface of positive Gaussian curvature and, in particular, if the radii of the shell at its ends are equal to zero, then the shell has the shape of an oblate ellipsoid of revolution. A shell of spherical shape corresponds to the value of the dimensionless parameter $\alpha = 1$. If $\alpha = 1$, the type of shell changes and, when $\alpha < 1$, the peripheral stress σ_θ is critical for the shapes which have been found. However, when α is reduced, the optimal shell continues to have a positive Gaussian curvature up to a value $\alpha = \alpha_1$. When $\alpha = \alpha_1$, the curvature becomes equal to zero and the optimal shell takes a conical shape. Then, when $\alpha_1 > \alpha > \alpha_0$, the optimal shell takes the shape of a convex-concave surface of revolution of negative Gaussian curvature. When $\alpha \to \alpha_0$, the surface of the optimal shell tends to a surface composed of conical surfaces and, moreover, at the point $x = x_0$, the radius of the shell $r(x_0) = 0$. The construction of the optimal shapes for the problem in question when $\alpha < \alpha_0$ leads to the search for solutions which have segments where $r = 0$, and they are omitted as not being of any physical interest.

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